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On the density of states of sparse random matrices

Yan V Fyodorov† and A D Mirlin

Leningrad Nuclear Physics Institute, 188350 Gatchina, Leningrad District, USSR

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Abstract. The supersymmetric method of calculation of density of states of a sparse random matrix is shown to be absolutely equivalent to the replica trick. A functional generalization of Hubbard–Stratonovich (HS) transformation was used in the course of the solution.

Our paper is concerned with an application of a supersymmetric approach to a problem of calculation of the density of states of a real, symmetric $N\times N$ matrix $\hat{H}$ $(N\to\infty)$ whose elements H_{ij} $(=H_{ji})$ are independent, identically distributed random variables, with a probability distribution

$$P(H_{ij})=\left(1-\frac{p}{N}\right)\delta(H_{ij})+\frac{p}{N}h(H_{ij}) \tag{1}$$

$h(z)$ being any even distribution function non-singular in $z=0$ and 'connectivity' p, i.e. mean number of non-zero elements per row, being of order of unity.

This problem was investigated in [1] by means of the replica trick, and an integral equation was obtained giving a basic possibility of finding the density of states. Authors of the recent paper [2] tried to use a supersymmetric method of calculation of density of states developed in [3]. They derived a system of integral equations that, from their point of view, had no simple relations to the analogous equation obtained in [1], and only for $p\to\infty$ was it demonstrated that both approaches give identical results.

We show that for the problem of calculation of density of states the supersymmetric approach turns out to be absolutely equivalent to the replica trick. We use a functional generalization of Hubbard–Stratonovich (HS) transformation, which from our point of view makes the derivation more transparent and general in comparison with the one used in [1, 2].

The density of states of a random matrix H_{ij} within the scope of a supersymmetric approach is given by the following expressions [3, 4]:

$$\rho(E)=(2\pi N)^{-1}\,\mathrm{Im}\frac{\partial}{\partial J}\langle Z(E,J)\rangle|_{J=0}$$

$$Z(E,J)=\int\prod_i[\mathrm{d}\phi_i]\exp\left\{\frac{\mathrm{i}}{2}\sum_{ij}\phi_i^+[(E\hat{I}+J\hat{K})\delta_{ij}-H_{ij}]\phi_j\right\} \tag{2}$$

† Present address: Fachbereich Physik, Universität Gesamthochschule Essen D-4300, Essen 1, Federal Republic of Germany.

where

$$\phi_i \begin{bmatrix} S_i^{(1)} \\ S_i^{(2)} \\ \chi_i \\ \chi_i^* \end{bmatrix} \qquad \phi_i^+ = (S_i^{(1)}, S_i^{(2)}, \chi_i^*, -\chi_i)$$

is a supervector with two real commutative components $S_i^{(1)}$, $S_i^{(2)}$ and two Grassmannian components χ_i, χ_i^*; $[\mathrm{d}\phi_i] = \mathrm{d}S_i^{(1)}\,\mathrm{d}S_i^{(2)}\,\mathrm{d}\chi_i^*\,\mathrm{d}\chi_i$,

$$\hat{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and $\hat{I}$ is the identity matrix. The angular brackets in equation (2) denote the averaging over distribution (1).

Performing the averaging, we get for $N \to \infty$

$$\langle Z(E, J)\rangle = \int \prod_i [\mathrm{d}\phi_i] \exp\left\{\frac{\mathrm{i}}{2}\sum_i \phi_i^+(E\hat{I} + J\hat{K})\phi_i + \frac{p}{2N}\sum_{ij}[\tilde{h}(\phi_i^+\phi_j) - 1]\right\} \tag{3}$$

$$\tilde{h}(z) = \int h(t)\exp(-\mathrm{i}tz)\,\mathrm{d}t$$

To decouple the variables ϕ_i connected with different sites i, the authors of [1, 2], following the method developed in [5], expanded a function $\tilde{h}$ as a power series, and decoupled every term by the introduction of auxiliary variables (HS transformation) in the usual way. The integration over the auxiliary variables could be performed for $N \to \infty$ by the steepest descent method resulting in an infinite set of coupled saddle-point equations. Introducing a generating function one succeeds in rewriting this set in terms of a single integral equation [1, 2, 5].

Instead of the above-mentioned procedure we suggest using a functional generalization of HS transformation in order to decouple different sites:

$$\exp\left\{\frac{p}{2N}\sum_{ij}[\tilde{h}(\phi_i^+\phi_j) - 1]\right\} = \int \mathrm{D}g \exp\left\{-\frac{Np}{2}\int [\mathrm{d}\psi][\mathrm{d}\psi'] g(\psi)C(\psi, \psi')g(\psi') + p\sum_i g(\phi_i)\right\} \tag{4}$$

where $C(\psi, \psi')$ is determined by the relation

$$\int [\mathrm{d}\chi] C(\phi, \chi)[\tilde{h}(\chi^+\eta) - 1] = \delta(\phi, \eta) \tag{5}$$

$\delta(\phi, \eta)$ being δ-function in the space of supervectors.

It can be proved that an integral operator with the kernel $\tilde{h}(\phi^+\chi) - 1$ can be inverted in the space of even functions $g(\phi)$ vanishing in the origin. We discuss this matter in more detail at the end of this section.

Using equation (4) we transform equation (3) to the form:

$$\langle Z(E, j)\rangle = \int \mathrm{D}g \exp\left\{-\frac{Np}{2}\int [\mathrm{d}\psi][\mathrm{d}\psi'] g(\psi)C(\psi, \psi')g(\psi') + N\ln\int [\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{2}\phi^+(E\hat{I} + J\hat{K})\phi + pg(\phi)\right]\right\}. \tag{6}$$

Performing the functional integration over g for $N\to\infty$ by the steepest descent method, we get (for $J=0$) the following saddle-point equation:

$$\int [\mathrm{d}\psi']C(\psi,\psi')g(\psi')=\frac{\int[\mathrm{d}\phi]\delta(\phi,\psi)\exp((\mathrm{i}/2)\phi^+E\phi+pg(\phi))}{\int[\mathrm{d}\phi]\exp((\mathrm{i}/2)\phi^+E\phi+pg(\phi))}. \tag{7}$$

Using (5) we obtain from (7) the integral equation for the function $g(\phi)$:

$$g(\psi)=\frac{\int[\mathrm{d}\phi]\{\tilde{h}(\phi^+\psi)-1\}\exp((\mathrm{i}/2)\phi^+E\phi+pg(\phi))}{\int[\mathrm{d}\phi]\exp((\mathrm{i}/2)\phi^+E\phi+pg(\phi))} \tag{8}$$

that coincides with equation (20) from [2].

In view of invariance of the (8) with respect to a transformation $g(\phi)\to g(\hat{T}\phi)$, $\hat{T}$ being an arbitrary unitary supermatrix, it is natural to search for its solution as a function $\tilde{g}$ of the invariant $\phi^+\phi=S^2+2\chi^*\chi$; $S^2=(S^{(1)})^2+(S^{(2)})^2$. In that case the denominator in (8) turns out to be unity (a general problem of an integration of invariant functions over supervectors was considered in details in [4]). Since $\tilde{g}(\phi^+\phi)=\tilde{g}(S^2)+2\chi^*\chi\tilde{g}'(S^2)$, equation (8) after the integration over Grassmannian components of the supervector ϕ is equivalent to two equations:

$$\tilde{g}(S^2)=-S\int_0^\infty \mathrm{d}R\exp\left[\frac{\mathrm{i}}{2}ER^2+p\tilde{g}(R^2)\right]\int \mathrm{d}z\, zh(z)J_1(zRS) \tag{9}$$

$$\tilde{g}'(S^2)=-\tfrac{1}{2}\int_0^\infty \mathrm{d}R\, R\exp\left[\frac{\mathrm{i}}{2}ER^2+p\tilde{g}(R^2)\right]\int \mathrm{d}z\, z^2h(z)J_0(zRS). \tag{10}$$

The density of states is related to the function $\tilde{g}(s^2)$ as follows:

$$\rho(E)=-\frac{2}{\pi B}\mathrm{Re}\,\tilde{g}'(0)\qquad B=\int \mathrm{d}z\, h(z)z^2. \tag{11}$$

A prime in equations (9)-(11) denotes a derivative of a function over its argument. It is easy to make sure that (10) is the direct consequence of (9); this supports correctness of the ansatz chosen for the function g. Equation (9) coincides with equation (18) of [1] for a special form of the distribution function $h(H_{ij})$ used therein.

Equation (9) was obtained [1] by the replica trick with an additional assumption that the solution to the problem is replica symmetric. In some way a supersymmetric approach could be considered as a specific variant of replica approach with half the replicated fields being anticommutative variables. From this point of view our assumption that g is a function of $\phi^+\phi$ only seems to be equivalent to replica symmetric ansatz. Unfortunately, we are unable to prove the absence of solutions without this symmetry, but we believe that it is the only 'symmetric' solution that is relevant for the problem under consideration.

The authors of [2] sought a solution of equation (8) in more general form

$$g(\phi)=A(S^2)+2\chi^*\chi B(S^2) \tag{12}$$

that results in three coupled equations for the functions A, B and $\tilde{Z}$, where $\tilde{Z}$ denotes the denominator on the right-hand side of (8). One can easily make sure that the condition $A'=B$ is consistent with that system of equations and reduces it to a single equation equivalent to equation (9).

Let us discuss now the question of the existence of the quantity C, defined as the kernel of the operator inverse to $\tilde{h}-1$ (see (5)). Such an inversion could be performed only in the absence of zero eigenvalues of the operator $\tilde{h}-1$. According to the definition, eigenfunctions f and eigenvalues λ of that operator satisfy the following equation:

$$\int [\mathrm{d}\psi]\{\tilde{h}(\phi^+\psi)-1\}f(\psi)=\lambda f(\phi). \tag{13}$$

Let us restrict ourselves to consideration of the functions that vanish at the origin†. We look for eigenfunctions in a general form

$$f(\phi)=f_1(S^{(1)},S^{(2)})+f_2(S^{(1)},S^{(2)})\chi^*\chi. \tag{14}$$

Performing the integration over Grassmanian variables χ^*, χ we reduce (13) to the system of integral equations:

$$\begin{aligned}
&-\int \frac{\mathrm{d}R^{(1)}\,\mathrm{d}R^{(2)}}{2\pi}[\tilde{h}(\boldsymbol{RS})-1]f_2(\boldsymbol{R})=\lambda f_1(\boldsymbol{S})\\
&-\int \frac{\mathrm{d}R^{(1)}\,\mathrm{d}R^{(2)}}{2\pi}\tilde{h}''(\boldsymbol{RS})f_1(\boldsymbol{R})=\lambda f_2(\boldsymbol{S}).
\end{aligned} \tag{15}$$

Taking into account the fact of invariance of this system with respect to O(2)-rotations we should look for its solution at the form $f_t(\boldsymbol{S})=f_t^{(m)}(S^2)\exp(\mathrm{i}m\phi_s)$, $t=1,2$, where ϕ_s is the polar angle of the vector $\boldsymbol{S}$; m-integer number.

Performing the integration over angular variable we get

$$\begin{aligned}
&-\mathrm{i}^m\int \mathrm{d}R\,R\int \mathrm{d}z\,h(z)[J_m(zRS)-\delta_{0m}]f_2^{(m)}(R^2)=\lambda f_1^{(m)}(S^2)\\
&-\mathrm{i}^m\int \mathrm{d}R\,R\int \mathrm{d}z\,h(z)J_m(zRS)f_1^{(m)}(R^2))=\lambda f_2^{(m)}(S^2).
\end{aligned} \tag{16}$$

Due to the fact that $h(z)=h(-z)$ the left-hand sides of equations (16) vanish for odd m and any $f_{1,2}^{(m)}(S^2)$, so $\lambda=0$. That means that we should consider the operator $\tilde{h}-1$ acting within the space of even functions $f(\boldsymbol{S})=f(-\boldsymbol{S})$.

For even $m\neq 0$ and special form of distribution function $h(z)=\frac{1}{2}[\delta(z-1)+\delta(z+1)]$ investigated in [1, 2], an integral transform in the left-hand side of equation (16) is nothing but the well known Hankel transform and we immediately find that $\lambda=\pm 1$. It is easy to show that $\lambda=\pm 1$ for $m=0$ as well. It is possible to prove the absence of zero modes for more general distribution functions $h(z)$ also, so it seems that an operator inverse to $\tilde{h}-1$ exists in the space of even functions vanishing in the origin for any reasonable distribution function $h(z)$.

In conclusion, we have shown that the supersymmetric approach and the replica trick both lead to the identical form of density of states of the large sparse random matrix.

However, as was shown in [6], the replica trick is inapplicable to a calculation of a correlation function of random matrix eigenvalues. The solution of this particular problem for sparse matrices within the scope of a supersymmetric approach is considered in a separate publication [7].

† Otherwise we should take into account the existence of δ-like zero mode: $f(\psi)=\delta(\psi^+\psi)$. We are grateful to Professor M R Zirnbauer for making us aware of this fact.

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